Math Logic: Model Theory & Computability
Lecture 19

Propositional / Boolean axioms. For all σ-formulas φ, ψ, η, the following are axioms:

(1) $\varphi \to (\psi \to \varphi)$

Remark. This is equivalent to $(\varphi \wedge \psi) \to \varphi$, which is obviously true in every structure.

(2) $(\varphi \to \psi) \to \Big((\varphi \to (\psi \to \eta)) \to (\varphi \to \eta) \Big)$.

Remark. This is equiv. to $\Big(\big[(\varphi \to \psi) \wedge ((\varphi \wedge \psi) \to \eta) \big] \wedge \varphi \Big) \to \eta$.

(3) Proof-by-contradiction: $(\neg \varphi \to \psi) \to \big[(\neg \varphi \to \neg \psi) \to \varphi \big]$.

Remark. This is equiv. to $\big[(\neg \varphi \to \psi) \wedge (\neg \varphi \to \neg \psi) \big] \to \varphi$.

Quantifier axioms. For all σ-formulas φ, ψ, η, for all variables v, and for all σ-terms t that are OK to plug-in for v in φ, the following are axioms:

(4) Instantiation: $(\forall v \varphi) \to \varphi(t/v)$.

(5) Generalization: $\varphi \to (\forall v \varphi)$.

Def. For an extended σ-formula $\varphi(\vec{x})$, we say that a σ-structure $\underline{A}$ satisfies φ, and write $\underline{A} \models \varphi$, if $\underline{A} \models \forall \vec{x} \varphi$. This explains Axiom (5).

Equality axioms.

(6) Equality is an equivalence relation: For all variables x, y, z, the following are axioms:

(6.a) Reflexivity: $x = x$.
(6.b) Symmetry: $(x = y) \to (y = x)$.
(6.c) Transitivity: $(x = y \land y = z) \to x = z$.

(7) Functions respect equality: For each k-ary $f \in \mathrm{Func}(\sigma)$ and k-tuples $\vec{x} := (x_1, \dots, x_k)$ and $\vec{y} := (y_1, \dots, y_k)$ of variables, the following is an axiom:

$$\vec{x} = \vec{y} \to f(\vec{x}) = f(\vec{y}),$$

where by $\vec{x} = \vec{y}$ we mean $x_1 = y_1 \land x_2 = y_2 \land \dots \land x_n = y_n$, and the whole formula above is an abbreviation for

$$x_1 = y_1 \to (x_2 = y_2 \to (\dots (\to x_n = y_n \to f(x_1, \dots, x_k) = f(y_1, \dots, y_k)).$$

(8) Relations respect equality: For each k-ary $R \in \mathrm{Rel}(\sigma)$ and k-tuples $\vec{x} := (x_1, \dots, x_k)$ and $\vec{y} := (y_1, \dots, y_k)$ of variables, the following is an axiom:

$$\vec{x} = \vec{y} \to (R(\vec{x}) \to R(\vec{y})).$$

Rule of inference: Modus Ponens: For all σ-formulas φ, ψ, the rule is:

$$\varphi, \ \varphi \to \psi \ \overset{MP}{\rightsquigarrow} \ \psi.$$

We would say that ψ is obtained by MP from φ and $\varphi \to \psi$.
E.g. All humans are mortal, Socrates is a human, therefore Socrates is mortal.

Obs. All axioms in Axiom(σ) hold in every σ-structure, and Modus Ponens preserves this.

Proof. Let $\underline{A} := (A, \sigma)$ be a σ-structure and let φ, ψ be σ-formulas. We show that (1) $\theta := (\varphi \to (\psi \to \varphi))$ holds in $\underline{A}$. Let $\vec{x} := (x_1, \dots, x_k)$ such that $\theta(\vec{x})$ is an extended formula. T show that it holds in $\underline{A}$, we fix an arbitrary $\vec{a} \in A^k$ and show that $\underline{A} \models \theta(\vec{a})$. Suppose $\underline{A} \models \varphi(\vec{a})$, then clearly $\underline{A} \models \psi(\vec{a}) \to \varphi(\vec{a})$ by the definition of interpretation of $\to$. (Indeed, $\eta \to \xi$ holds, by definition, if ξ holds or η fails, i.e. $\neg\eta \vee \xi$). The proofs for the rest of the axioms are similar, and clearly Modus Ponens preserves satisfiability again by the def. of interpretation of $\to$.

Def. Let T be a σ-theory and φ be a σ-formula. A (formal) proof from T is a finite sequence $(\varphi_1, \varphi_2, \dots, \varphi_n)$ of σ-formulas such that for each $i = 1, \dots, n$,

- either $\varphi_i \in \text{Axiom}(\sigma) \cup T$
- or φ_i is obtained by Modus Ponens (MP) from φ_j and φ_k for some $j, k < i$, in particular $\varphi_k := \varphi_j \to \varphi_i$.

We say that T proves φ, denoted $T \vdash \varphi$, if there is a proof $(\varphi_1, \dots, \varphi_n)$ from T with $\varphi_n = \varphi$. If $T = \emptyset$, we would just write $\vdash \varphi$, and if $T = T' \cup \{\eta_1, \dots, \eta_e\}$, we would write $T', \eta_1, \dots, \eta_e \vdash \varphi$.

Obs (Soundness of the proof system). For a σ-theory T and a σ-formula φ,

$$\text{if } T \vdash \varphi \text{ then } T \models \varphi.$$

Proof. Let $(\varphi_1, \dots, \varphi_n)$ be a proof of φ from T. Then we show by induction on i that $T \models \varphi_i$. But this follows from the previous observation that Axioms(σ) is satisfied by all σ-structures, T is satisfied by all models of T, and Modus Ponens preserves satisfiability. $\square$

Examples of formal proofs.

Prop. Let σ be a signature. For all σ-formulas φ, ψ:

(a) $\varphi \vdash \psi \to \varphi$.
(b) $\vdash \varphi \to \varphi$.

Proof. (a) (1) Axiom (1): $\vdash \varphi \to (\psi \to \varphi)$.

(2) $\varphi \vdash \varphi$

(3) MP (2),(1): $\varphi \vdash \psi \to \varphi$.

(b) (1) Axiom (2) with $\varphi := \theta$, $\psi := \theta \to \theta$, $\eta := \theta$: $\vdash (\theta \to (\theta \to \theta)) \to [(\theta \to ((\theta \to \theta) \to \theta)) \to (\theta \to \theta)]$.

(2) Axiom (1): $\vdash \theta \to (\theta \to \theta)$

(3) MP (2),(1): $\vdash (\theta \to ((\theta \to \theta) \to \theta)) \to (\theta \to \theta)$

(4) Axiom (1): $\vdash \theta \to ((\theta \to \theta) \to \theta)$

(5) MP (4),(3): $\vdash \theta \to \theta$. $\square$